Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On Some Cases of Motion of a Solid in Infinite Liquid.

By G. Kolosoff.

The present paper contains some elliptic-function solutions of the motion under no force of a solid in infinite liquid of the same character as considered by Mr. A. G. Greenhill in Vol. XX and XXVIII of the American Journal of Mathematics, but for a body of different external shape.

(I) The external shape of the body is that considered by Clebsch in Mathematische Annalen III with the form of the kinetic energy*

$$2T = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + b_1y_1^2 + b_2y_2^2 + b_3y_3^2$$

where

$$a_1\left(\frac{1}{b_2} - \frac{1}{b_3}\right) + a_2\left(\frac{1}{b_3} - \frac{1}{b_1}\right) + a_3\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0. \qquad (1)$$

With the six dynamical equations:

$$\frac{dx_1}{dt} = -x_3\frac{\partial T}{\partial y_2} + x_2\frac{\partial T}{\partial y_3}, \quad \frac{dy_1}{dt} = -x_3\frac{\partial T}{\partial x_2} + x_2\frac{\partial T}{\partial x_3} - y_3\frac{\partial T}{\partial y_2} + y_2\frac{\partial T}{\partial y_3}$$

. . . .

we arrive immediately at the three integrals:

$$\left.\begin{array}{ll} 2T = const. = l & x_1y_1 + x_2y_2 + x_3y_3 = const. = m \\ x_1^2 + x_2^2 + x_3^2 = const. = n & \end{array}\right\} \qquad (2)$$

and the fourth integral of Clebsch:

$$b_1a_1y_1^2 + b_2a_2y_2^2 + b_3a_3y_3^2 - (a_2a_3x_1^2 + a_3a_1x_2^2 + a_1a_2x_3^2) = const. = k. \qquad (3)$$

In particular cases $m = 0$ and $b_1 = b_2 = b_3$ this motion is expressible by the double-theta hyperelliptic function.† The most general case was reduced by F. Kötter to the case $b_1 = b_2 = b_3$, but his investigation contained mistakes as shown by the investigations of the Russian mathematicians A. Liapounoff and W. Stekloff.*

---

* The notation employed is that of the paper of Mr. A. G. Greenhill.

† H. Weber, Mathematische Annalen XIV. F. Kötter, Crelle, t. 109.

Let us suppose that the constants in the expression of the kinetic energy satisfy besides (1) also the conditions:

$$\frac{1}{b_3} = \frac{1}{b_1} + \frac{1}{b_2}\dagger \quad (b_3 < b_1 \text{ and } b_3 < b_2).$$

Then $(a_3 - a_2)\, b_1 = (a_3 - a_1)\, b_2$ and $(a_1 - a_2)\, b_2 = (a_3 - a_2)(b_2 - b_1)$.
From the integrals (2) and (3) we have:

$$b_2 (a_3 - a_1)(x_1^2 + x_2^2) - (b_2^2 y_2^2 + b_1^2 y_1^2) = \frac{b_2}{a_3 - a_2}\Big(la_3 - k - n(a_3^2 + a_1 a_2)\Big).$$

Let us suppose

$$la_3 = k + n(a_3^2 + a_1 a_2).$$

The differential equations admit then a fractional integral:

$$\frac{b_1 x_1 y_1 + b_2 x_2 y_2}{(a_3 - a_2)\, x_2^2 - b_1 y_1^2} = const. = \Gamma. \tag{3*}$$

Besides this integral the fundamental integrals give us the integral

$$x_3 - \frac{a_3 - a_2}{a_2 - a_1}\Gamma\, y_3 = const. = \Gamma_1. \tag{4}$$

Bearing in mind that

$$\varepsilon^2 (x_1^2 + x_2^2)^2 - (b_1 x_1 y_1 + b_2 x_2 y_2)^2 = (b_2 y_2 x_1 - b_1 y_1 x_2)^2$$

where

$$\varepsilon = +\sqrt{b_2 (a_3 - a_1)} = +\sqrt{b_1 (a_3 - a_2)},$$

and taking into consideration the integral

$$x_1^2 + x_2^2 + x_3^2 = const. = n$$

we deduce

$$\left(\frac{dx_3}{dt}\right)^2 = (b_2 y_2 x_1 - b_1 y_1 x_2)^2 = \varepsilon^2 (n - x_3^2)^2 - \Gamma^2\Big((a_3 - a_2)\, x_2^2 - b_1 y_1^2\Big)^2. \tag{5}$$

---

* W. Stekloff. On the motion of solid in infinite liquid. Charcow, 1893 (in Russian language).

† The cases $\frac{1}{b_1} = \frac{1}{b_2} + \frac{1}{b_3}$ and $\frac{1}{b_2} = \frac{1}{b_1} + \frac{1}{b_3}$ can be treated similarly.

But from the integrals (2) and (3) we deduce

$$(a_1 - a_2)\, b_2 x_2^2 + (b_1 - b_2)\, b_1 y_1^2 = b_2 b_3 y_3^2 + \Gamma_2, \text{ where } \Gamma_2 = n b_2 a_3 - b_2 k,$$

and (5) takes the form

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2 (n - x_3^2)^2 - \Gamma^2 \frac{(b_2 b_3 y_3^2 + \Gamma_2)^2}{(b_2 - b_1)^2}, \tag{6}$$

since

$$(a_3 - a_2)(b_2 - b_1) = (a_1 - a_2)\, b_2 \tag{7}$$

and

$$(a_3 - a_2)\, x_2^2 - b_1 y_1^2 = \frac{(a_1 - a_2)\, b_2}{b_2 - b_1} x_2^2 - b_1 y_1^2 = \frac{1}{b_2 - b_1}\Big((a_1 - a_2)\, b_2 x_2^2 + b_1 (b_1 - b_2) y_1^2\Big)$$

$$= \frac{1}{b_2 - b_1}(b_2 b_3 y_3^2 + \Gamma_2),$$

and from (4), (6):

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2 (n - x_3^2)^2 - \frac{\Gamma^2}{(b_2 - b_1)^2}(b_2 b_3 y_3^2 + \Gamma_2)^2 =$$

$$= \varepsilon^2 \left\{ (n - x_3^2)^2 - \frac{\Gamma^2}{(b_2 - b_1)^2 \varepsilon^2}\left(b_2 b_3 \frac{(a_2 - a_1)^2}{\Gamma^2 (a_3 - a_2)^2} (x_3 - \Gamma_1)^2 + \Gamma_2 \right)^2 \right\},$$

and therefore

$$\int \frac{dx_3}{\sqrt{(n - x_3^2)^2 - \frac{\Gamma^2}{(b_2 - b_1)^2 \varepsilon^2}\left(b_2 b_3 \frac{(a_2 - a_1)^2}{\Gamma^2 (a_3 - a_2)^2} (x_3 - \Gamma_1)^2 + \Gamma_2 \right)^2}} = \varepsilon t + \mathit{const.},$$

or

$$\int \frac{dx_3}{\sqrt{(n - x_3^2)^2 - (\alpha + \beta x_3 + \gamma x_3^2)^2}} = \varepsilon t + \mathit{const.},$$

where α, β, γ are constants.

From this we can conclude that the problem may be reduced to elliptic functions.

I have shown* that the motion in this case can be resolved into two motions à la Poinsot and that the elements of the motion can be expressed by Weierstrassian Sigma Functions.

---

* In my Russian paper "On Some Modifications of the Hamilton's Principle" this solution is developed and discussed. A short account of this paper is given by Lampe in "Forschritte der Mathematik," 1903.

As individual cases of this theory we may regard and reduce to elliptic functions the following cases:

$$\begin{array}{lll}
(A) & y_1b_1 = \varepsilon x_2\,, & y_2b_2 = \varepsilon x_1 \\
(B) & y_1b_1 = -\varepsilon x_2\,, & y_2b_2 = -\varepsilon x_1 \\
(C) & y_1b_1 = \varepsilon x_1\,, & y_2b_2 = -\varepsilon x_2 \\
(D) & y_1b_1 = -\varepsilon x_1\,, & y_2b_2 = \varepsilon x_2 .
\end{array}$$

We may deduce these solutions immediately from the six differential equations. without using the integral (3*). Let us suppose (A) $y_1b_1 = \varepsilon x_2$, $y_2b_2 = \varepsilon x_1$. The first three differential equations of the problem take the form:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= b_3y_3x_2 - \varepsilon x_1x_3 \\
\frac{dx_2}{dt} &= \varepsilon x_2x_3 - b_3y_3x_1 \\
\frac{dx_3}{dt} &= \varepsilon(x_1^2 - x_2^2)
\end{aligned}\right\} \qquad (8)$$

The 6th equation gives

$$y_3 = const. = \omega\,,$$

the 4th and the 5th become

$$\frac{dx_2}{dt} = \frac{b_1(a_3 - a_2)}{\varepsilon}\,x_2x_3 + \frac{(b_3 - b_2)b_1}{b_2}\,x_1y_3\,,$$

$$\frac{dx_1}{dt} = \frac{b_2(a_1 - a_3)}{\varepsilon}\,x_3x_1 + \frac{(b_1 - b_3)b_2}{b_1}\,x_2y_3\,,$$

and coincide with the first two equations (8), by (7) and by the relation

$$(a_3 - a_1)b_2 = (a_3 - a_2)b_1 = \varepsilon^2.$$

The integrals (2) take the form

$$x_1^2 + x_2^2 + x_3^2 = n$$

$$x_1x_2 + \frac{b_3}{\varepsilon}\,\omega x_3 = \frac{mb_3}{\varepsilon}\,,$$

and we have

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2(n - x_3^2)^2 - 4b_3^2(m - \omega x_3)^2.$$

Therefore the problem is reduced to elliptic functions. In the same manner we can investigate the case B. Let us suppose (C) $y_1 b_1 = \varepsilon x_1 \quad y_2 b_2 = -\varepsilon x_2$. The differential equations of motion take the form:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= b_3 y_3 x_2 + \varepsilon x_2 x_3 \\ \frac{dx_2}{dt} &= \varepsilon x_1 x_3 - b_3 y_3 x_1 \\ \frac{dx_3}{dt} &= -2\varepsilon x_1 x_2 \\ \frac{dy_3}{dt} &= 2(a_2 - a_1) x_1 x_2 \end{aligned}\right\}. \tag{9}$$

The other two equations coincide with (9) in the same way as in case A.

The integrals (2) and (3) are reduced to

$$\left.\begin{aligned} & x_1^2 + x_2^2 + x_3^2 = n \\ & \frac{\varepsilon x_1^2}{b_1} - \frac{\varepsilon x_2^2}{b_2} + x_3 y_3 = m \\ & x_1^2 - x_2^2 + \frac{b_3}{a_1 - a_2} y_3^2 = \frac{l - a_3 n}{a_1 - a_2} = const. = \Gamma_0 \end{aligned}\right\}. \tag{10}$$

Instead of one of these integrals we can take the integral

$$x_3(a_1 - a_2) - y_3 \varepsilon = const. = \Gamma_1{}^* \tag{11}$$

which follow at once from the differential equations (9).
Then

$$\frac{dx_3}{dt} = \varepsilon \sqrt{(n - x_3^2)^2 - \left(\Gamma_0 - \frac{b_3}{a_1 - a_2} y_3^2\right)^2},$$

and with (11) the problem is reduced to elliptic functions.

The case D can be investigated in the same way as C.

(II). As is probably already known, lately the Russian mathematicians, W. Stekloff and A. Liapounoff discovered two new integrable cases of motion of a solid in infinite liquid (under no force) which A. Clebsch did not discover because of an error in his calculations.

In these cases the problem may be also reduced to ultra elliptic functions,

---

$$* \; \Gamma_1 = (a_1 - a_2) \sqrt{n + \Gamma_0 \frac{b_2 + b_1}{b_2 - b_1} - 2m \frac{\varepsilon}{a_1 - a_2}}.$$

but in particular cases to simple elliptic functions. In Stekloff's case the kinetic energy is expressed by the formula:

$$2T = \Sigma\, \tau^2 a_1 (a_2^2 + a_3^2)\, x_1^2 + 2\tau a_2 a_3 x_1 y_1 + a_1 y_1^2$$

and the differential equations appear as follows:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= a_3 x_2 y_3 - a_2 x_3 y_2 - x_2 x_3 \tau a_1 (a_3 - a_2)\\
\frac{dx_2}{dt} &= a_1 x_3 y_1 - a_3 x_1 y_3 - x_3 x_1 \tau a_2 (a_1 - a_3)\\
\frac{dx_3}{dt} &= a_2 x_1 y_2 - a_1 x_2 y_1 - x_1 x_2 \tau a_3 (a_2 - a_1)\\
\frac{dy_1}{dt} &= y_2 y_3 (a_3 - a_2) - \tau a_1 (a_3 - a_2)(y_3 x_2 + x_3 y_2) +\\
&\qquad + \tau^2 (a_2 - a_3)(a_2 a_3 - a_1^2)\, x_2 x_3\\
\frac{dy_2}{dt} &= y_3 y_1 (a_1 - a_3) - \tau a_2 (a_1 - a_3)(y_1 x_3 + x_1 y_3) +\\
&\qquad + \tau^2 (a_3 - a_1)(a_3 a_1 - a_2^2)\, x_3 x_1\\
\frac{dy_3}{dt} &= y_1 y_2 (a_2 - a_1) - \tau a_3 (a_2 - a_1)(y_2 x_1 + x_2 y_1) +\\
&\qquad + \tau^2 (a_1 - a_2)(a_1 a_2 - a_3^2)\, x_1 x_2
\end{aligned}\right\} \tag{12}$$

and admit a 4th algebraical integral of W. Stekloff:

$$\Sigma\, \tau^2 a_1 (a_1 - 2a_2 - 2a_3)\, x_1^2 + 2\tau a_1 x_1 y_1 - y_1^2 = const.$$

These differential equations admit a particular solution

$$y_1 = \tau (a_2 - a_3)\, x_1 \qquad y_2 = \tau (a_1 - a_3)\, x_2{}^{*} \tag{13}$$

since the equations (1), (2), (4), (5) of the system (12) can be reduced by (13) to two differential equations

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= a_3 x_2 y_3 + \tau a_3 (a_2 - a_1)\, x_2 x_3\\
\frac{dx_2}{dt} &= -\, a_3 x_1 y_3 + \tau a_3 (a_2 - a_1)\, x_1 x_3\\
&\text{and the equations (3) and (6) give}\\
\frac{dx_3}{dt} &= 2\tau x_1 x_2 a_3 (a_1 - a_2)\\
\frac{dy_3}{dt} &= 2\tau^2 (a_2 - a_1)\, a_3 (2a_3 - a_1 - a_2)\, x_1 x_2
\end{aligned}\right\}. \tag{14}$$

---

[*] In the same manner can be treated the solution: $y_2 = \tau (a_3 - a_1)\, x_2$ $y_3 = \tau (a_2 - a_1)\, x_3$ and the solution $y_3 = \tau (a_1 - a_2)\, x_3$ $y_1 = \tau (a_3 - a_2)\, x_1$.

The system (14) admits the integrals:

$$
\begin{aligned}
& x_1^2 + x_2^2 + x_3^2 = n \\
& x_1^2 - x_2^2 = \frac{1}{\tau^2 (a_2 - a_1)(2a_3 - a_1 - a_2)} y_3^2 + \Gamma \\
& x_3 \tau (2a_3 - a_1 - a_2) + y_3 = \Gamma_1
\end{aligned}
\qquad (15)
$$

where n, Γ, Γ_1 are the constants of integration and we obtain:

$$
\frac{dx_3}{dt} = \tau a_3 (a_1 - a_2) \sqrt{(n - x_3^2)^2 - \left(\Gamma + \frac{1}{\tau^2 (a_2 - a_1)(2a_3 - a_1 - a_2)} y_3^2\right)^2}
$$

and by (15) the problem is reduced to elliptic functions. The motion in this case can be also resolved into two motions à la Poinsot.

In Liapounoff's case the kinetic energy is expressed by the formula:

$$
2T = \Sigma b y_1^2 + 2c_1 x_1 y_1 + \frac{(c_2 - c_3)^2}{b} x_1^2,
$$

and we obtain the differential equations of motion in the form:

$$
\left.
\begin{aligned}
\frac{dx_1}{dt} &= b(x_2 y_3 - y_2 x_3) + (c_3 - c_2) x_2 x_3 \\
\frac{dx_2}{dt} &= b(x_3 y_1 - y_3 x_1) + (c_1 - c_3) x_3 x_1 \\
\frac{dx_3}{dt} &= b(x_1 y_2 - y_1 x_2) + (c_2 - c_1) x_1 x_2 \\
\frac{dy_1}{dt} &= (c_3 - c_2)\left\{x_3 y_2 + x_2 y_3 + \frac{x_2 x_3}{b}(2c_1 - c_2 - c_3)\right\} \\
\frac{dy_2}{dt} &= (c_1 - c_3)\left\{x_1 y_3 + x_3 y_1 + \frac{x_2 x_1}{b}(2c_2 - c_3 - c_1)\right\} \\
\frac{dy_3}{dt} &= (c_2 - c_1)\left\{x_2 y_1 + x_1 y_2 + \frac{x_1 x_2}{b}(2c_3 - c_1 - c_2)\right\}
\end{aligned}
\right\}
\qquad (16)
$$

and admit besides the three known integrals (2) the integral of A. Liapounoff:

$$
\Sigma c_1 ((c_2 + c_3) x_1 + b y_1)^2 = const.
$$

The differential equations (16) admit a particular solution:

$$y_1 = \frac{c_3 - c_2}{b} x_1 \qquad y_2 = \frac{c_3 - c_1}{b} x_2 .^{*} \tag{17}$$

As before, in Stekloff's case, we reduce the system (16) by (17) to four equations:

$$\frac{dx_1}{dt} = bx_2y_3 - (c_2 - c_1)x_2x_3$$

$$\frac{dx_2}{dt} = -(c_2 - c_1)x_1x_3 - bx_1y_2$$

$$\frac{dx_3}{dt} = 2(c_2 - c_1)x_1x_2$$

$$\frac{dy_3}{dt} = 2(c_2 - c_1)\,\frac{2c_3 - c_1 - c_2}{b}\,x_1x_2 .$$

The integrals of this system of differential equations we obtain in the form

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= n \\ x_1^2 - x_2^2 + \frac{b^2}{(c_1 - c_2)\,(2c_3 - c_1 - c_2)}\,y_3^2 &= \Gamma \\ y_3 + \frac{c_2 + c_1 - 2c_3}{b}\,x_3 &= \Gamma_1 \end{aligned} \tag{18}$$

and we find

$$\frac{dx_2}{dt} = (c_2 - c_1)\sqrt{(n - x_3^2)^2 - \left(\Gamma - \frac{b_2}{(c_1 - c_2)\,(2c_3 - c_1 - c_2)}\,y\right)^2} .$$

By (18) the problem is, as before, reduced to elliptic functions. The motion in this case can also be resolved into two motions à la Poinsot. These solutions are connected with some pseudo-elliptic integrals.

For instance let us consider the solution A of the case I. We have:

$$y_1b_1 = \varepsilon x_2 \quad y_2b_2 = \varepsilon x_1 \quad y_3 = const. = \qquad \omega x_1^2 + x_2^2 + x_3^2 = n$$

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2(n - x_3^2)^2 - 4b_3^2(m - \omega x_3)^2 .$$

---

[*] The solution $y_2 = \frac{c_1 - c_3}{b} x_2$, $y_3 = \frac{c_1 - c_2}{b} x_3$ and the solution $y_3 = \frac{c_2 - c_1}{b} x_3$, $y_1 = \frac{c_2 - c_3}{b} x_1$ can be treated in the same manner.

Let us suppose that the axis Z fixed in space coincides with the axis of impulse and the position of the axes of x, y, z fixed in the body is given by the three Euler's angles θ, ϕ, ψ. Let p, q, r be the components of angular velocity in the directions of the axes of x, y, z. Then

$$\sin^2\theta\frac{d\psi}{dt}=\frac{1}{\sqrt{n}}(px_1+qx_2)$$

$$\frac{d\phi}{dt}=r-\cos\theta\frac{d\psi}{dt}.$$

But

$$p=\frac{\partial T}{\partial y_1}=b_1y_1=\varepsilon x_2,\quad q=\frac{\partial T}{\partial y_2}=b_2y_2=\varepsilon x_1,\quad r=\frac{\partial T}{\partial y_3}=b_3\omega$$

therefore

$$\frac{d\phi}{dt}=b_3\omega-\frac{2\varepsilon x_1x_2}{n-x_3^2}=b_3\frac{\omega n-2x_3m+\omega x_3^2}{n-x_3^2}.$$

But

$$\frac{1}{\sqrt{n}}(x_1+ix_2)=-\sin\theta\, e^{-i\phi}$$

so that

$$\phi=-\frac{1}{i}\,lg\,\frac{x_1+ix}{\sqrt{n_2-x_3^2}}.$$

We can conclude from this that the integral

$$\int\frac{\omega n-2x_3m+\omega x_3^2}{n-x_3^2}\,\frac{dx_3}{\sqrt{\varepsilon^2(n-x_3^2)^2-4b_3^2(m-\omega x_3)^2}}$$

is pseudo elliptic and

$$=-\frac{1}{b_3i}\,lg\,\frac{x_1+ix_2}{\sqrt{n-x_3^2}}=$$

$$=-\frac{1}{2ib_3}\,lg\,\frac{\frac{1}{\epsilon}\sqrt{\varepsilon^2(n-x_3^2)^2-4b_3^2(m-\omega x_3)^2}+\frac{2ib_3}{\epsilon}(m-\omega x_3)}{\sqrt{n-x_3^2}}.$$

This result can be verified by differentiating; and similar pseudo-elliptic integrals can be written also in the other cases.

Regarding investigations of Mr. Roger Liouville* I very much doubt that he really found a new integrable case of motion of a solid in infinite liquid as in the case of the problem of rotation of a heavy body about a fixed point, where he professed to have found a new integrable case. On the strength of which statement this case was entered under his name in different text books.*

---

* Comptes rendus, 1896, II sém.

In 1898 my attempts to find integrals of the 8th and 12th power according to his method failed completely, as did also the attempts of Prof. S. Tshapliguine in Moscow. Whenever I succeeded in getting an integral according to Liouville's method it always turned out to be simply a reduction of already known integrals and only gave a new integral in the case of Mme. Sophie Kovalevsky. In the just published (January, 1906) article by Husson in the "Annales de Toulouse" this theorem is generally proved and so the integrable case of R. Liouville of rotation of a heavy body about a fixed point no longer exists.

JURJEW (Dorpat), Russia.

---

* Appell. Traité de mécanique rationelle, *t* II.